Some reported properties of Lambert W functions. As was remarked on page 9, Lambert W functions are mentioned in none of the standard methods of mathematical physics texts, treated in none of the standard higher function handbooks, though the web now provides many sources, of which for me the most valuable have been the Wikipedia article "Lambert W Function" and the paper by Corless *et al* cited there. In particular, the Lambert W function is not indexed in Abramowitz & Stegun's *Handbook of Mathematical Functions*, a 1965 publication of the National Bureau of Standards.

Yesterday Peter Renz, my friend of more than half a century, brought to my belated attention the facts that NIST (National Institute of Standards & Technology, successor to the NBS) has published a successor to Abramowitz & Stegun, which is available on the web as the DLMF = Digital Library of Mathematical Functions. And that §4.13 in DLMF does (*i*) summarized basic properties of the Lambert W function, and (*ii*) provide essential references. It was a couple of the formulæ reported there that led me to return to this subject.

The Wikipedia article reports that asymptotically ($x \to \infty$)

$$\begin{aligned} W_0(x) &= L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2+L_2)}{2L_1^2} + \frac{L_2(6-9L_2+2L_2^2)}{6L_1^2} + \cdots \\ &= L_1 - L_2 + \sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty} \frac{(-)^\ell}{m!}\begin{bmatrix} \ell+m \\ \ell+1 \end{bmatrix} L_1^{-\ell-m} L_2^m \end{aligned}$$

where $L_1 = \log(x)$, $L_2 = \log(\log(x))$ and the $\begin{bmatrix} \ell+m \\ \ell+1 \end{bmatrix}$ are non-negative Stirling numbers of the first kind. If we write $\xi = \log x$ this becomes

$$W_0(x) = \xi - \log\xi + \frac{\log\xi}{\xi} + \Big[\frac{(\log\xi)^2}{2\xi^2} - \frac{\log\xi}{\xi^2}\Big] + \cdots \tag{15}$$

as reported in DLMF.

Of greater interest to me—because, as remarked on page 10, *Mathematica* responds to the command `Series[LambertW`$_{-1}$`[`xe^x`],{x,0,10}]` with "a very long and unworkable mess"—is the DLMF report that if we write $\eta = \log(-1/x)$ then, as $0 \uparrow x$,

$$W_{-1}(x) = -\eta - \log\eta - \frac{\log\eta}{\eta} - \Big[\frac{(\log\eta)^2}{2\eta^2} + \frac{\log\eta}{\eta^2}\Big] + \cdots \tag{16}$$

which is a sharpened version of a rather vague statement encountered in the Wikipedia article.

That article also supplies this continued fraction, taken from a Russian paper (2006):

$$\begin{aligned} W_0(x) = x/(1+(x/(1+(x/(2+(5x/(3+(17x/(10+ \\ (133x/(17+(1927x/(190+(13582711x/(94423+\cdots \end{aligned} \tag{17}$$

An analogous development of $W_{-1}(x)$ is not reported; if it exists it must not be confused with Sommerfeld's iterative process, which he notated

$$\exp[W_{-1}(x)] = \cfrac{x}{\log \cfrac{x}{\log \cfrac{x}{\log \cfrac{x}{\cdots}}}}$$

and allowed himself to call a "continued fraction," which it isn't.

Figure 11 shows that the asymptotic expression on the right-hand side of (15) is, though truncated, remarkably accurate already at $x = 10$; the

$$\text{relative error} = \frac{W_0^{\text{asymptotic}}(x) - W_0(x)}{W_0(x)}$$

climbs to 0.0027 at $x = 35$ and thereafter falls monotonically (Figure 12). We expect the relative error to be still further reduced when higher order terms are incorporated into the construction of $W_0^{\text{asymptotic}}(x)$.

The success of (16) is less dramatic (Figure 13), but the relative error is seen in Figure 14 to vanish as $-0.002 < x \uparrow 0$; *i.e.*, as $W_{-1}(x) \downarrow -\infty$. We note in this connection that Sommerfeld had interest in the values assumed by

$$S(z) \equiv \exp[W_{-1}(z)] \quad : \quad z = x + iy \tag{18.1}$$

when x is a very small negative number, and y a very small positive number; he was led from the physics of his problem[19] to consider $z = -1.44(1-i)10^{-7}$ to be typical. One might, on the basis of (16), suppose that

$$S(z) \approx \exp\Big\{ -\eta - \log\eta - \frac{\log\eta}{\eta} - \Big[\frac{(\log\eta)^2}{2\eta^2} + \frac{\log\eta}{\eta^2}\Big]\Big\} \tag{18.2}$$

where again $\eta = \log(-1/z)$. At z_{typical} that gives

$$S(z_{\text{typical}}) \approx (7.27243 - 7.93001i) \times 10^{-9}$$

whereas (18.1) gives a result

$$S(z_{\text{typical}}) \approx (7.49012 - 8.20184i) \times 10^{-9}$$

that is produced also—already in sixth order—by the the P-process of page 13 (Sommerfeld's recursive algorithm).

The remarkable precision of the (truncated) continued fraction (17) is evident in Figure 15. The curves cannot be distinguished at this resolution; the associated relative error is shown in Figure 16.

[19] See page 28 of the Warnock translation of "On the propagation of electrodynamic waves along a wire."